Physics 523, Quantum Field Theory II

Homework 1

Due Wednesday, 14th January 2004

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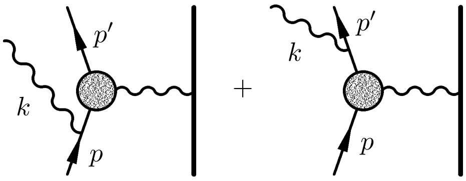
Bremsstrahlung

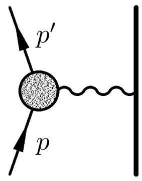
a) We showed that in the low energy limit, the amplitude for Bremsstrahlung,

$$i\widetilde{\mathcal{M}} =$$



$$= e\overline{u}(p')\mathcal{M}_o(p',p)u(p)\left(\frac{p'^\mu}{p'\cdot k} - \frac{p^\mu}{p\cdot k}\right)\epsilon^*_\mu, \tag{1.1}$$

can be written in terms of the amplitude for the process without bremsstrahlung which given in terms of the relativistically corrected amplitude $\mathcal{M}_o(p',p)$,



$$= i\overline{u}(p')\mathcal{M}_o(p',p)u(p).$$

We are to verify that (1.1) does indeed vanish when $\epsilon_\mu = k_\mu$. This can be easily seen by direct calculation.

$$i\widetilde{\mathcal{M}} = e\overline{u}(p')\mathcal{M}_o(p',p)u(p)\left(\frac{p'^\mu}{p'\cdot k} - \frac{p^\mu}{p\cdot k}\right)k_\mu,$$

$$= e\overline{u}(p')\mathcal{M}_o(p',p)u(p)\left(\frac{p'^\mu k_\mu}{p'\cdot k} - \frac{p^\mu k_\mu}{p\cdot k}\right),$$

$$= e\overline{u}(p')\mathcal{M}_o(p',p)u(p)\left(1-1\right),$$

$$\boxed{\therefore i\widetilde{\mathcal{M}}^\mu k_\mu = 0.} \tag{1.2}$$

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b) While in the soft photon limit this amplitude is consistent with current conservation, we will show that it fails in complete generality. To see this, let us consider the full amplitude for the two diagrams,

$$i\mathcal{M} = e\overline{u}(p')\left\{\mathcal{M}_o(p',p-k)\frac{\not{p}-\not{k}+m}{(p-k)^2-m^2}\gamma^\mu\epsilon^*_\mu(k) + \gamma^\mu\epsilon^*_\mu\frac{\not{p}'+\not{k}+m}{(p'+k)^2-m^2}\mathcal{M}_o(p'+k,p)\right\}u(p). \tag{1.3}$$

Now, recalling our work with the Dirac equation (and its conjugate) we see that,

$$(\not{p}+m)\gamma^\mu u(p) = 2p^\mu u(p), \quad \text{and} \quad \overline{u}(p')\gamma^\mu(\not{p}'+m) = \overline{u}(p')2p'^\mu.$$

Combining this result with simple kinematics for the case where $\epsilon_\mu = k_\mu$ we have

$$i\mathcal{M} = e\overline{u}(p')\left\{k_\mu\frac{2p'^\mu+\gamma^\mu\not{k}}{2p'\cdot k}\mathcal{M}_o(p'+k,p) - \mathcal{M}_o(p',p-k)\frac{2p^\mu-\not{k}\gamma^\mu}{2p\cdot k}k_\mu\right\}u(p),$$

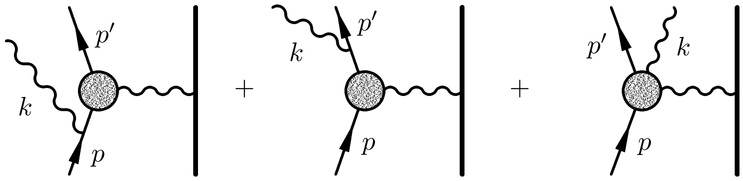
$$= e\overline{u}(p')\left\{\frac{2p'\cdot k+\not{k}\not{k}}{2p'\cdot k}\mathcal{M}_o(p'+k,p) - \mathcal{M}_o(p',p-k)\frac{2p\cdot k-\not{k}\not{k}}{2p\cdot k}\right\}u(p),$$

$$= e\overline{u}(p')\left[\mathcal{M}_o(p'+k,p) - \mathcal{M}_o(p',p-k)\right]u(p),$$

Now, this result cannot be vanishing for an arbitrary photon energy k. It is certainly the case that $\mathcal{M}_o(p'+k,p) = \mathcal{M}_o(p',p-k)$ to the order $\mathcal{O}(1/k)$ but certainly not in general. We will have to add an additional diagram to see true current conservation.

c) We can improve our estimate of the amplitude to emit a photon by Bremsstrahlung by adding a third diagram in which the photon is emitted from the 'gut' of the reaction with some amplitude $i\mathcal{M}_3 = e\overline{u}(p')\epsilon^*_\mu S^\mu u(p)$. Adding this diagram, we arrive have

$i\mathcal{M}_{\text{total}} =$



$$= \epsilon^*_\mu(k)e\overline{u}(p')\left\{\frac{2p'^\mu + \gamma^\mu \not{k}}{2p'\cdot k}\mathcal{M}_o(p'+k,p) - \mathcal{M}_o(p',p-k)\frac{2p^\mu - \not{k}\gamma^\mu}{2p\cdot k} - S^\mu\right\}u(p). \tag{1.4}$$

Therefore, we see that gauge invariance which demands that $k_\mu \mathcal{M}^\mu_{\text{total}} = 0$ implies that

$$k_\mu \mathcal{M}^\mu_{\text{total}} = 0 = e\overline{u}(p')k_\mu\left\{\frac{2p'^\mu + \gamma^\mu \not{k}}{2p'\cdot k}\mathcal{M}_o(p'+k,p) - \mathcal{M}_o(p',p-k)\frac{2p^\mu - \not{k}\gamma^\mu}{2p\cdot k} - S^\mu\right\}u(p),$$
$$= e\overline{u}(p')[\mathcal{M}_o(p'+k,p) - \mathcal{M}_o(p',p-k) - k_\mu S^\mu]u(p),$$

Therefore we see at once that gauge invariance implies that

$$k_\mu S^\mu = \mathcal{M}_o(p'+k,p) - \mathcal{M}_o(p',p-k). \tag{1.5}$$

d) Let us expand in derivatives of $\mathcal{M}_o$'s on the right. Doing this, we see that

$$k_\mu S^\mu = \frac{\partial}{\partial p'^\mu}\mathcal{M}_o(p',p)k^\mu + \frac{\partial}{\partial p^\mu}\mathcal{M}_o(p',p)k^\mu. \tag{1.6}$$

This implies that

$$S^\mu = \left(\frac{\partial}{\partial p'_\mu} + \frac{\partial}{\partial p_\mu}\right)\mathcal{M}_o(p',p) + \text{ divergenceless term.}$$

Now, At low energy, all divergenceless terms will go to zero and so our approximation of

$$S^\mu = \left(\frac{\partial}{\partial p'_\mu} + \frac{\partial}{\partial p_\mu}\right)\mathcal{M}_o(p',p), \tag{1.7}$$

is good to $\mathcal{O}(1)$.

Returning to the process of soft Bremsstrahlung, we see that the total amplitude to order $\mathcal{O}(1)$ can be written as

$$i\mathcal{M}_{\text{total}} = e\overline{u}(p')u(p)\left\{\frac{p'^\mu}{p'\cdot k} - \frac{p^\mu}{p\cdot k} - \frac{\partial}{\partial p'_\mu} - \frac{\partial}{\partial p_\mu}\right\}\epsilon^*_\mu(k)\mathcal{M}_o(p',p)u(p). \tag{1.8}$$